JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **31**, No. 3, August 2018
http://dx.doi.org/10.14403/jcms.2018.31.1.343

THE COMPLETE CONVERGENCE FOR DEPENDENT RANDOM VARIABLES IN HILBERT SPACES

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Abstract. We study the complete convergence for sequences of dependent random variables in Hilbert spaces. Results are obtained for negatively associated random variables and ϕ-mixing random variables in Hilbert spaces.

1. Introduction

A finite family of random variables $\{X_i, 1 \leq i \leq n\}$ is said to be negatively associated (NA) if for every pair of disjoint nonempty subsets A and B of $\{1, 2, \cdots, n\}$ and any real coordinatewise nondecreasing functions f on $\mathbb{R}^{|A|}$, g on $\mathbb{R}^{|B|}$,

$$Cov(f(X_i, i \in A), g(X_j, j \in B)) \leq 0, \tag{1.1}$$

whenever f and g are such that the covariance exists. Here and in the sequel $|A|$ denotes the cardinality of A. An infinite family of random variables is NA if every finite subfamily is NA. The concept of negative association for random variables was introduced by Joag-Dev and Proschan in [7].

The concept of negative association was extended to finite dimensional space and to Hilbert space (for details see Zhang [12] and Ko *et al.* [8]).

Ko *et al.* [8] introduced the concept of negative association for $\mathbb{R}^d$-valued random variables as follows: A finite family of $\mathbb{R}^d$-valued random variables $\{X_i, 1 \leq i \leq n\}$ is said to be NA if for every pair of disjoint nonempty subsets A and B of $\{1, 2, \cdots, n\}$ and any real coordinatewise nondecreasing functions f on $\mathbb{R}^{|A|d}$, g on $\mathbb{R}^{|B|d}$, (1.1) is satisfied.

Received May 29, 2018; Accepted 2018.
2010 Mathematics Subject Classification: 60F15, 60G10.
Key words and phrases: complete convergence, negative associated random variables, ϕ-mixing random variables, Hilbert space, weakly upper bounded.
This paper was supported by the Sehan University Research Fund in 2018.

Let H be a real separable Hilbert space with the norm $\|\cdot\|$ generated by an inner product $< \cdot, \cdot >$ and $\{e_j, j \geq 1\}$ be an orthonormal basis in H. Let $< X, e_j >$ be denoted by $X^{(j)}$ for an H-valued random variables X.

Ko *et al.* [8] extended the concept of negative association in $\mathbb{R}^d$ to Hilbert space as follows.

A sequence $\{X_n, n \geq 1\}$ of H-valued random variables is said to be negatively associated(NA) if for some orthonormal basis $\{e_j, j \geq 1\}$ of H and for any $d \geq 1$, the sequence $\{(X_n^{(1)}, X_n^{(2)}, \cdots, X_n^{(d)}), n \geq 1\}$ of $\mathbb{R}^d$-valued random variables is NA.

In many papers one can find some interesting results concerning sequences of H-valued negatively associated random variables. We refer only some of them.

Almost sure convergence by Ko *et al.* [8]. almost sure convergence extending the results of *Ko et al.* [8] by Thanh [11], Hájek-Rényi inequality by Miao [10], Baum-Katz type theorem for the case $r > \frac{1}{\alpha}$, by Huan *et al.* [5], complete convergence for the case $r = \frac{1}{\alpha}$, and weak laws of large numbers by Hien and Thanh [3].

Let $\{X, X_n, n \geq 1\}$ be a sequence of H-valued random variables. We consider the following inequalities

$$C_1 P(\|X\| > t) \leq \frac{1}{n} \sum_{k=1}^{n} P(\|X_k\| > t) \leq C_2 P(\|X\| > t), \tag{1.2}$$

for all $t > 0$.

If there exists a positive constant C_1 (C_2) such that left-hand side (right-hand side) of (1.2) is satisfied for all $n \geq 1$ and $t \geq 0$, then the sequence $\{X_n, n \geq 1\}$ is said to be weakly lower (upper) bounded by H-valued random vector X. The sequence $\{X_n, n \geq 1\}$ is said to be weakly bounded by X if it is both weakly lower and upper bounded by X (See [4]).

Note that (1.2) is, of course, automatic with $X = X_1$ and $C_1 = C_2 = 1$ if $\{X_n, n \geq 1\}$ is a sequence of identically distributed H-valued random variables.

In the rest of the paper, the symbol C will denote a generic positive constant which is not necessarily the same in each appearance.

In this paper we consider the Baum-Katz result for sequences H-valued dependent random variables.

2. Some lemmas

We introduce some useful notations and extend the lemma given by Gut [2] to Hilbert space.

Let us put

$$X_i^{'} = X_i(\|X_i\| \leq A),\ X_i^{''} = X_i I(\|X_i\| > A),$$

and

$$X^{'} = XI[\|X\| \leq A],\ X^{''} = XI(\|X\| > A),$$

for some constant $A > 0$.

LEMMA 2.1. *Let $\{X_n, n \geq 1\}$ be a sequence of H-valued random variables which are weakly bounded by a random variables X. Let $s > 0$.*

(2.1-1) *If $E\|X\|^s < \infty$, then $\frac{1}{n}\sum_{k=1}^n E\|X_k\|^s \leq CE\|X\|^s$,*
(2.1-2) $\frac{1}{n}\sum_{k=1}^n E\|X_k^{'}\|^s \leq C(E\|X^{'}\|^s + A^s P(\|X\| > A))$,
(2.1-3) $\frac{1}{n}\sum_{k=1}^n E\|X_k^{''}\|^s \leq CE\|X^{''}\|^s$.

Proof. The proof is based on the well-known fact that for any random variables Y with $E\|Y\|^s < \infty$

$$E\|Y\|^s = s\int_0^\infty y^{s-1}P(\|Y\| > y)dy.$$

This completes the proof. □

Next we consider the moment maximal inequality for H-valued NA random variables.

LEMMA 2.2. [5] *Let $\{X_n, n \geq 1\}$ be a sequence of H-valued NA random variables with $EX_n = 0$ and $E\|X_n\|^2 < \infty$, $n \geq 1$. Then we have*

$$E\max_{1\leq k\leq n}\|\sum_{i=1}^k X_i\|^2 \leq 2\sum_{i=1}^n E\|X_i\|^2,\ \ n \geq 1. \tag{2.2}$$

Let us note that there is a misprint in Lemma 3.3 of Ko *et al.* in [8].

3. Negatively associated H-valued random variables

THEOREM 3.1. *Let $\{X_n, n \geq 1\}$ be a sequence of H-valued negatively associated random variables with zero means. Let $\alpha r > 1$, $\alpha > \frac{1}{2}$ and*

$0 < r < 1$. *Suppose that* $\{X_n, n \geq 1\}$ *is weakly upper bounded by a random variables* X. *If*

$$E\|X\|^r < \infty, \tag{3.1}$$

then

$$\sum_{n=1}^{\infty} n^{\alpha r-2} P(\max_{1\leq k\leq n} \|S_k\| > \epsilon n^{\alpha}) \text{ for every } \epsilon > 0, \tag{3.2}$$

where $S_k = \sum_{l=1}^{k} X_l$.

Proof. For $l, n \geq 1$, set

$$X_l = X_l I[\|X_l\| \leq n^{\alpha}] + X_l I[\|X_l\| > n^{\alpha}],$$

and

$$S_n' = \sum_{l=1}^{n} X_l I[\|X_l\| \leq n^{\alpha}] + \sum_{l=1}^{n} X_l I[\|X_l\| > n^{\alpha}].$$

Then, by Chebyshev's inequality, (1.2) and Lemma 2.1(2.1-2), we have (3.3)

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha r-2} P(\max_{1\leq k\leq n} \|\sum_{l=1}^{k} X_l I[\|X_l\| \leq n^{\alpha}]\| > \epsilon n^{\alpha}) \\
&\leq C \sum_{n=1}^{\infty} n^{\alpha r-2-\alpha} E(\max_{1\leq l\leq n} \|\sum_{l=1}^{k} X_l I[\|X_l\| \leq n^{\alpha}]\|) \\
&\leq C \sum_{n=1}^{\infty} n^{\alpha r-2-\alpha} \sum_{l=1}^{n} E(\|X_l\| I[\|X_l\| \leq n^{\alpha}]\|) \\
&\leq C \sum_{n=1}^{\infty} n^{\alpha r-1-\alpha} E(\|X\| I[\|X\| \leq n^{\alpha}]\|) + C \sum_{n=1}^{\infty} n^{\alpha r-1} P(\|X\| > n^{\alpha}) \\
&= I_1 + I_2.
\end{aligned}$$

Then

$$\begin{aligned}
I_1 &= C \sum_{n=1}^{\infty} n^{\alpha r-1-\alpha} E(\|X\| I[(l-1)^{\alpha} < \|X\| \leq l^{\alpha}]) \\
&\leq C \sum_{n=1}^{\infty} n^{\alpha r-1-\alpha} l^{\alpha} P((l-1)^{\alpha} < \|X\| \leq l^{\alpha}]) \qquad (3.4) \\
&\leq C \sum_{l=1}^{\infty} l^{\alpha+\alpha r-\alpha} P((l-1)^{\alpha} < \|X\| \leq l^{\alpha}])
\end{aligned}$$

$$\begin{aligned}&\leq C\sum_{l=1}^{\infty} l^{\alpha r}P((l-1)^{\alpha}<\|X\|\leq l^{\alpha}])\\&=CE\|X\|^{r}<\infty \ \text{ by (3.1)}.\end{aligned}$$

Next we estimate that

$$\begin{aligned}I_2&=C\sum_{n=1}^{\infty} n^{\alpha r-1}P(\|X\|>n^{\alpha})\\&\leq C\sum_{n=1}^{\infty} n^{\alpha r-1}\sum_{l=n}^{\infty} P(l^{\alpha}<\|X\|\leq (l+1)^{\alpha})\\&\leq C\sum_{l=1}^{\infty} l^{\alpha r}P(l^{\alpha}<\|X\|\leq (l+1)^{\alpha})\\&=E\|X\|^{r}<\infty \text{ by (3.1)}.\end{aligned}\tag{3.5}$$

By (3.3), (3.4) and (3.5) we obtain

$$\sum_{n=1}^{\infty} n^{\alpha r-2}P(\max_{1\leq k\leq n}\|\sum_{l=1}^{k} X_l I[\|X_l\|\leq n^{\alpha}]\|>\epsilon n^{\alpha})<\infty.\tag{3.6}$$

Similarly, we will prove that

$$\sum_{n=1}^{\infty} n^{\alpha r-2}P(\max_{1\leq k\leq n}\|\sum_{l=1}^{k} X_l I[\|X_l\|> n^{\alpha}]\|>\epsilon n^{\alpha})<\infty.$$

Using Markov inequality, (1.2) and Lemma 2.1 (2.1-3) we have

$$\begin{aligned}&\sum_{n=1}^{\infty} n^{\alpha r-2}P(\max_{1\leq k\leq n}\|\sum_{l=1}^{k} X_l I[\|X_l\|> n^{\alpha}]\|>\epsilon n^{\alpha})\\&\leq C\sum_{n=1}^{\infty} n^{\alpha r-2-\frac{\alpha r}{2}}E(\max_{1\leq k\leq n}\|\sum_{l=1}^{k} X_l I[\|X_l\|> n^{\alpha}]\|)^{\frac{r}{2}}\\&\leq C\sum_{n=1}^{\infty} n^{\alpha r-2-\frac{\alpha r}{2}}\sum_{l=1}^{n} E(\|X_l\|^{\frac{r}{2}} I[\|X_l\|> n^{\alpha}]\|)\ (\frac{r}{2}<\frac{1}{2})\\&\leq C\sum_{n=1}^{\infty} n^{\frac{\alpha r}{2}-1}E(\|X\|^{\frac{r}{2}} I[\|X\|> n^{\alpha}]\|)\\&\leq C\sum_{n=1}^{\infty} n^{\frac{\alpha r}{2}-1}\sum_{l=n}^{\infty} E(\|X\|^{\frac{r}{2}} I[l^{\alpha}<\|X\|\leq (l+1)^{\alpha}])\end{aligned}\tag{3.7}$$

$$\begin{aligned}
&\le C\sum_{n=1}^{\infty} n^{\frac{\alpha r}{2}-1}\sum_{l=n}^{\infty}(l+1)^{\frac{\alpha r}{2}}P(l^{\alpha}<\|X\|\le(l+1)^{\alpha})\\
&\le C\sum_{l=1}^{\infty} l^{\alpha r}P(l^{\alpha}<\|X\|\le(l+1)^{\alpha})\\
&= CE|X|^{r}<\infty.
\end{aligned}$$

Thus by (3.6) and (3.7) we obtain (3.2) and the proof of Theorem 3.1 is complete. □

COROLLARY 3.2. *Let $\{X_n, n \ge 1\}$ be a sequence of H-valued negatively associated and identically X-distributed random variables with mean zeros. Let $\alpha r > 1$, $\alpha > \frac{1}{2}$ and $0 < r < 1$. If $E\|X\|^r < \infty$, then (3.2) holds.*

COROLLARY 3.3. *Let $\{X_n, n \ge 1\}$ be a sequence of independent H-valued random vectors with mean zeros. Let $\alpha r > 1$, $\alpha > \frac{1}{2}$ and $0 < r < 1$. Suppose that $\{X_n, n \ge 1\}$ is weakly upper bounded by a random variables X. Then, (3.1) implies (3.2).*

THEOREM 3.4. *Let $\frac{1}{2} < \alpha < 1$ and let $\{X_n, n \ge 1\}$ be a sequence of H-valued NA random variables with mean zeros. Suppose that $\{X_n, n \ge 1\}$ is weakly bounded by a random variables X. Then (3.1) implies*

$$\sum_{n=1}^{\infty}\frac{1}{n}P(\max_{1\le k\le n}\|\sum_{i=1}^{k}X_i\|>\epsilon n^{\alpha})<\infty \text{ for all } \epsilon>0. \tag{3.8}$$

Proof. Let

$$X_k = X_k I(\|X_k\|\le n^{\alpha}) + X_k I(\|X_k\|>n^{\alpha}).$$

Then, for every $\epsilon > 0$ we have

$$\begin{aligned}
&\sum_{n=1}^{\infty}\frac{1}{n}P(\max_{1\le k\le n}\|\sum_{i=1}^{k}X_i\|>\epsilon n^{\alpha})\\
&\quad\le \sum_{n=1}^{\infty}\frac{1}{n}P(\max_{1\le k\le n}\|\sum_{i=1}^{k}X_iI(\|X_i\|\le n^{\alpha})\|>\frac{\epsilon}{2}n^{\alpha})\\
&\qquad+\sum_{n=1}^{\infty}\frac{1}{n}P(\max_{1\le k\le n}\|\sum_{i=1}^{k}X_iI(\|X_i\|>n^{\alpha})\|>\frac{\epsilon}{2}n^{\alpha})\\
&\quad= I_1+I_2.
\end{aligned}$$

Then

$$
\begin{aligned}
I_1 &= \sum_{n=1}^{\infty} \frac{1}{n} P(\max_{1 \le k \le n} \| \sum_{l=1}^{k} X_l I(\|X_l\| \le n^{\alpha}) \| > \frac{\epsilon}{2} n^{\alpha}) \\
&\le C \sum_{n=1}^{\infty} n^{-\alpha-1} E(\max_{1 \le l \le n} \| \sum_{l=1}^{k} X_l I(\|X_l\| \le n^{\alpha})) \\
&\le C \sum_{n=1}^{\infty} n^{-\alpha-1} \sum_{l=1}^{n} E(\|X_l\| I(\|X_l\| \le n^{\alpha})) \\
&\le C \sum_{n=1}^{\infty} n^{-\alpha-1} E(\|X\| I(\|X\| \le n^{\alpha})) + C \sum_{n=1}^{\infty} P(\|X\| > n^{\alpha}) \\
&= I_{11} + I_{12}.
\end{aligned}
$$

Here

$$
\begin{aligned}
I_{11} &= C \sum_{n=1}^{\infty} n^{-\alpha} E(\|X\| I(\|X\| \le n^{\alpha})) \\
&= C \sum_{n=1}^{\infty} n^{-\alpha} \sum_{l=1}^{n} E(\|X\| I[(l-1)^{\alpha} < \|X\| \le l^{\alpha}]) \\
&\le C \sum_{n=1}^{\infty} n^{-\alpha} \sum_{l=1}^{n} l^{\alpha} P[(l-1)^{\alpha} < \|X\| \le l^{\alpha}] \\
&\le C \sum_{l=1}^{\infty} l P[(l-1)^{\alpha} < \|X\| \le l^{\alpha}] \\
&= C E \|X\|^{\frac{1}{\alpha}} < \infty,
\end{aligned}
$$

and

$$
\begin{aligned}
I_{12} &= C \sum_{n=1}^{\infty} P(\|X\| > n^{\alpha}) \\
&= C \int_0^{\infty} P(\|X\| > x^{\alpha}) dx \\
&= C \frac{1}{\alpha} \int_0^{\infty} y^{\frac{1}{\alpha}-1} P(\|X\|^{\frac{1}{\alpha}} > y)\ dy \ (\text{letting } y = x^{\frac{1}{\alpha}}) \\
&= C E \|X\|^{\frac{1}{\alpha}} < \infty.
\end{aligned}
$$

Similarly, we prove that

$$I_2 = \sum_{n=1}^{\infty} \frac{1}{n} P(\max_{1 \le k \le n} \| \sum_{l=1}^{k} X_l I(\|X_l\| > n^{\alpha}) \|) < \infty,$$

because

$$\begin{aligned}
I_2 &\le C \sum_{n=1}^{\infty} n^{-1-\frac{\alpha r}{2}} E(\max_{1 \le k \le n} \| \sum_{l=1}^{k} X_l I(\|X_l\| > n^{\alpha}) \|)^{\frac{r}{2}} \\
&\le C \sum_{n=1}^{\infty} n^{-1-\frac{\alpha r}{2}} (\sum_{l=1}^{n} E\|X_l\| I(\|X_l\| > n^{\alpha}) \|)^{\frac{r}{2}} \\
&\le C \sum_{n=1}^{\infty} n^{-1-\frac{\alpha r}{2}} \sum_{l=1}^{n} E(\|X_l\|^{\frac{r}{2}} I(\|X_l\| > n^{\alpha})) \\
&\le C \sum_{n=1}^{\infty} n^{-\frac{\alpha r}{2}} E(\|X_l\|^{\frac{r}{2}} I(\|X_l\| > n^{\alpha})) \\
&\le C \sum_{n=1}^{\infty} n^{-\frac{\alpha r}{2}} \sum_{l=n}^{\infty} E(\|X_l\|^{\frac{r}{2}} I(l^{\alpha} < \|X_l\| \le (l+1)^{\alpha})) \\
&\le C \sum_{n=1}^{\infty} l P(l^{\alpha} < \|X_l\| \le (l+1)^{\alpha}) \\
&= CE\|X\|^{\frac{1}{\alpha}} < \infty.
\end{aligned}$$

This completes the proof. □

4. A sequence of ϕ-mixing random variables

Let $\{X_i, i \ge 1\}$ be a sequence of random variables and for any $1 \le i \le j \le \infty$ denote M_i^j as the σ-field generated by $\{X_k, i \le k \le j\}$. A sequence of random variables is said to be ϕ-mixing, if for any $A \in M_1^k$ and $B \in M_{k+j}^{\infty}$,

$$|P(B|A) - P(B)| \le \phi(j), \ \phi(j) \ge 0,$$

where $1 \ge \phi(1) \ge \phi(2) \ge \cdots$, and $\lim_{j \to \infty} \phi(j) = 0$. For more information of ϕ-mixing (see [1]). Intuitively, $\{X_1, X_2, \cdots, X_n\}$ is ϕ-mixing if X_i and X_{i+j} become virtually independent as j becomes large. For example, the waiting time W_i of an $M/M/1$ delay-in-queue is ϕ-mixing because W_i and W_{i+j} become virtually independent as j becomes large.

In addition, m-dependent sequence implies ϕ-mixing, while for gaussian processes, ϕ-mixing corresponds to m-dependence (see [6]).

A sequence of random vectors $\{X_n, n \geq 1\}$ with values in a separable real Hilbert space $(H, < \cdot, \cdot >)$ is said to be as ϕ-mixing, if some othonormal basis $\{e_k, k \geq 1\}$ of H and for any $d \geq 1$ the d-dimensional sequence $(< X_i, e_1 >, \cdots, < X_i, e_d >), i \geq 1$ is ϕ-mixing.

LEMMA 4.1. [9] *Let $\{X_n, n \geq 1\}$ be a sequence of ϕ-mixing random variables with finite second moments, zero means and $\sum_n \phi^{\frac{1}{2}}(n) < \infty$. Then there exists a positive constant C such that*

$$E \max_{1\leq k\leq n} (\| \sum_{i=1}^{k} X_i \|)^2 \leq C \sum_{i=1}^{n} E\|X_i\|^2.$$

Based on Lemma 4.1 and by using the same proofs as NA sequence we can also obtain some estimates and limit behaviors for ϕ-mixing sequence, which are similar to the results in Section 3.

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